The Prismoidal Correction Revisited

R. E. Deakin[1] and M. N. Hunter[2]

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Abstract: In earthwork volume computations the *Prismoidal* and *End-Area* formulae are often used. The End-Area formula is simpler and requires less field measurements so it is often the formula of choice. But, for certain solids, it overestimates the volume where the Prismoidal formula would, if used, give the correct volume. In such cases, End-Area volumes can be corrected by applying *Prismoidal Corrections*, and this is a common practice. This paper aims to show that this practice is only correct for certain solids – a fact not often stated in engineering surveying textbooks.

Introduction

In earthwork volume computations, for example road construction, railroad embankments and cuttings, dam construction, etc., the design is set-out in the field, cross-section information obtained at regular intervals perpendicular to a centre-line and volumes computed from the cross-section areas and the interval distances.

A general assumption about the solid between the cross-sections is that it is a *prismoid* – a solid having parallel plane end-faces, not necessarily similar nor having the same number of edges, and with plane side-faces extending the full length of the solid (see Figure 1). Prismoids may be decomposed into the basic geometric solids; prisms, pyramids and wedges, and the volume of a prismoid is obtained from the *Prismoidal* formula

$$V_P = \frac{L}{6}\left(A_1 + 4A_m + A_2\right) \tag{1}$$

A_1, A_2 are the areas of the parallel end-faces, A_m is the area of the mid-section and L is the perpendicular distance between the end-faces. A derivation of this formula is given below.

A simple formula for estimating volumes of solids is the *End-Area* formula

$$V_{EA} = \frac{L}{2}\left(A_1 + A_2\right) \tag{2}$$

A_1, A_2 are the areas of the parallel end-faces and L is the perpendicular distance between the end-faces.

If the solid is a prismoid composed of prisms and wedges, the End-Area formula will give the correct volume, i.e., a volume as would be obtained using the Prismoidal formula. But, as will be demonstrated later, if the prismoid is composed of prisms, wedges and pyramids; wedges and pyramids; or pyramids only, the End-Area formula may under estimate or over estimate the correct volume. This under or over estimation is due to the presence of pyramids.

In earthwork volume computations for road construction, a common practice is to compute volumes (assuming solid prismoids) using the End-Area formula, and realizing that these solids may contain prisms, wedges and pyramids, apply *Prismoidal Corrections* to obtain volumes that would have been obtained if the Prismoidal formula had been used. This practice is attractive as it requires a minimum of field measurements, since no mid-section areas are required, and the usual formula for the Prismoidal Correction is simple. But as will be demonstrated, this practice is only correct for certain types of solid sections. And if the actual section differs from the assumed section then the Prismoidal Correction will be incorrect. This can be a problem if practitioners are using surveying engineering software that applies Prismoidal Corrections to End-Area volumes and they enter field information related to solids that are not the type applicable to the particular Prismoidal Correction.

In addition to the discussion of the appropriateness or otherwise of Prismoidal Corrections, this paper also provides some information on the use of the Prismoidal formula when the solids may contain curved surfaces.

The Prismoid and Newton's proof of the Prismoidal Formula

Figure 1 shows a prismoid. A_1, A_2 are the areas of the end-faces, A_m is the area of the mid-section and L is the perpendicular distance between end-faces. Note that the mid-section is parallel with the end-faces but its area is not necessarily the mean of A_1 and A_2. The volume

[1] School of Mathematical and Geospatial Sciences, RMIT University, GPO Box 2476V, Melbourne, VIC 3001, Australia. E-mail: rod.deakin@rmit.edu.au
[2] Maribyrnong VIC 3032, Australia.

is given by the Prismoidal formula shown above as equation (1).

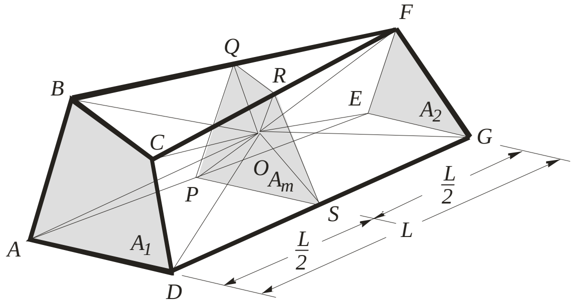


Figure 1: A prismoid

Our definition of a prismoid, which is common in surveying texts, is different from that used by mathematicians, who define our prismoid as a *prismatoid* (Weisstein 2008).

The Prismoidal formula is a computational formula dating from antiquity and appears on one of the oldest documents in existence, a papyrus scroll (about 544 centimetres long and 8 centimetres wide), written in Egypt around 1890BC. This papyrus scroll commonly known as the *Moscow Papyrus* – or *Golenischev Papyrus* after the Russian V. S. Golenischev who purchased it in Egypt in 1893 and sold it to the Moscow Museum of Fine Arts, where it still resides – contains 25 mathematical problems with solutions. The 14th problem asks for the volume of a truncated pyramid (frustum) and its stated solution can be expressed in the common form we know as the Prismoidal formula.

Interesting historical information regarding the Prismoidal formula in the Moscow Papyrus can be found on the Internet e.g., *The Prismoidal Formula* (Math Pages 2008) and *Moscow Mathematical Papyrus* (Wikipedia 2008). For those interested in the history of mathematics, Newman (1956) has a wonderful description of the *Rhind Papyrus;* another ancient Egyptian scroll describing fundamental mathematical principles.

The verification of the Prismoidal formula set out below, was enunciated by Sir Isaac Newton (1642-1726) and can be found in Clark (1957). It is interesting to note that Newton held the view – outlandish at the time – that he and others were just re-discovering the knowledge of the ancient Egyptians.

In Figure 1, let *PQRS* represent the section of area A_m midway between the end-faces *ABCD* and *EFG* and parallel to them. Take any point *O* in the plane of the mid-section and join *O* to the vertices of both end-faces. The prismoid is thus divided into a number of pyramids, each having its apex at *O,* and the bases of these pyramids form the end- and side-faces of the prismoid. The volume of the two pyramids whose bases are the end-faces are, respectively

$$\frac{A_1}{3}\times\frac{L}{2}=\frac{L}{6}A_1 \quad \text{and} \quad \frac{A_2}{3}\times\frac{L}{2}=\frac{L}{6}A_2$$

To express the volume of the pyramids based on the side-faces of the prismoid, consider, say, pyramid *OADGE*, and let the perpendicular distance of *O* from *SP* be *h*, then the volume of the pyramid *OADGE* is

$$\frac{1}{3}\ \text{area}(ADGE)\times h=\frac{1}{3}\left(PS\times L\times h\right)=\frac{2L}{3}\times\Delta\ OPS$$

where $\Delta\ OPS$ denotes the area of triangle *OPS.*

In the same manner, the volume of pyramid *OCDGF* = $\frac{2L}{3}\times\Delta\ ORS$ and so on for the others, so that the volume of the prismoid is given by

$$\begin{aligned} V &= \frac{L}{6}A_1+\frac{L}{6}A_2 \\ &\quad +\frac{2L}{3}\left(\Delta OPS+\Delta ORS+\Delta OQR+\Delta OPQ\right) \\ &= \frac{L}{6}A_1+\frac{L}{6}A_2+\frac{2L}{3}A_m \\ &= \frac{L}{6}\left(A_1+4A_m+A_2\right) \qquad (3) \end{aligned}$$

Shepherd (1983) shows how the Prismoidal formula can be applied to the solids: *cone, sphere, frustum of a cone,* and the *wedge* to yield the formula for the volume of each solid. Following his examples it can also be applied to the *ellipsoid, pyramid* and *frustum of a pyramid*.

Estimation problems using the End-Area formula

Figures 2, 3 and 4 show prismoids that are composed of, respectively; a prism and a wedge; a prism, a wedge and a pyramid; and two pyramids. In Figure 5, the prismoid is the result of removing four pyramids from a prism.

The volumes computed using the Prismoidal formula and shown in Table 1 are correct and can be verified by calculating the volumes of the composite prisms, wedges and pyramids; or the volume of an enclosing prism minus the volumes of pyramids and wedges.

In each of these figures, the prismoid has a length $L=10$ m and a rectangular base 2 m × 10 m with end-faces perpendicular to the base. The areas of the end-faces and mid-sections are easily obtained and volumes computed using equations (1) and (2), the Prismoidal and End-Area formulas respectively.

Prismoid	Areas m^2			Volumes m^3	
	A_1	A_m	A_2	V_P	V_{EA}
Figure 2	6	5.0	4	50	50
Figure 3	8	6.5	4	$63\frac{1}{3}$	60
Figure 4	2	2.5	4	$26\frac{2}{3}$	30
Figure 5	3	3.0	2	$28\frac{1}{3}$	25

Table 1: Volumes of prismoids

For example in Figure 3 the volumes of the prism, wedge and pyramid are

$$V_{\text{PRISM}} = \text{end area} \times \text{perpendicular height} = 40 \text{ m}^3$$

$$V_{\text{WEDGE}} = \frac{1}{6}\left(\begin{array}{l}\text{sum of parallel edges}\\ \times \text{ base width}\\ \times \text{ perpendicular height}\end{array}\right) = 10 \text{ m}^3$$

$$V_{\text{PYRAMID}} = \frac{1}{3}\left(\begin{array}{l}\text{end area}\\ \times \text{ perpendicular height}\end{array}\right) = 13\tfrac{1}{3} \text{ m}^3$$

and the volume of the prismoid is

$$V_{\text{PRISMOID}} = V_{\text{PRISM}} + V_{\text{WEDGE}} + V_{\text{PYRAMID}} = 63\tfrac{1}{3} \text{ m}^3$$

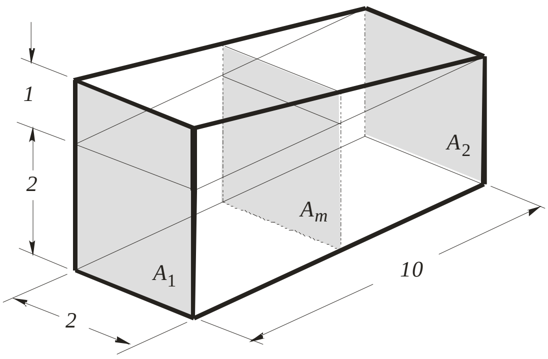


Figure 2: Prism+wedge

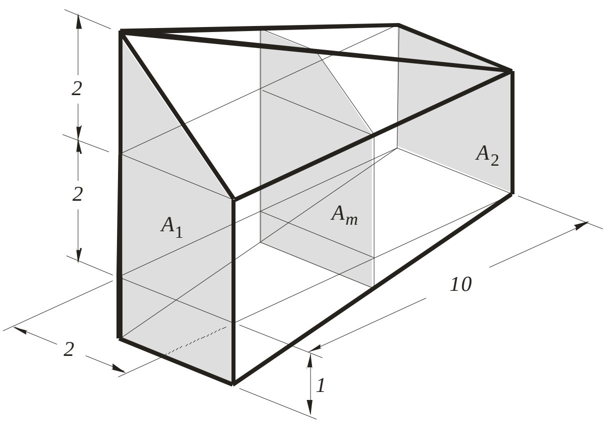


Figure 3: Prism+wedge+pyramid

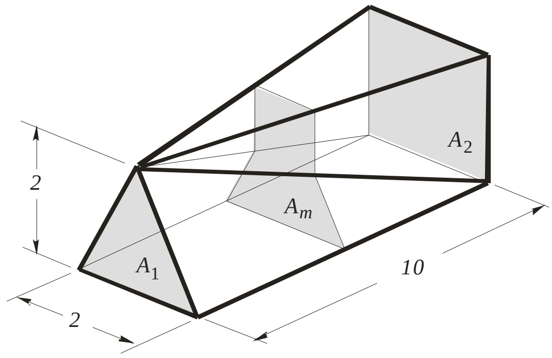


Figure 4: Pyramids

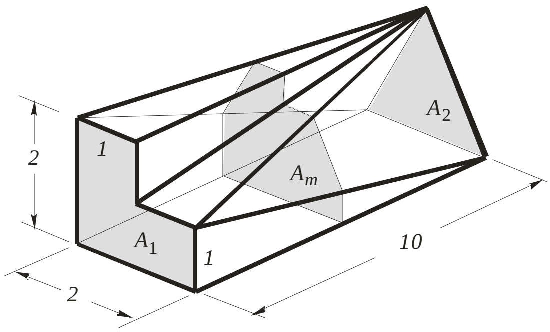


Figure 5: Prism–pyramids

We may infer, from the examples presented, that volumes computed by the End-Area formula are sometimes equal to, sometimes less than and sometimes greater than volumes computed using the Prismoidal formula. Equality occurs if, and only if, the mid-section area is the arithmetic mean of the end-areas, i.e., if $A_m = \frac{1}{2}\left(A_1 + A_2\right)$ and in such cases, the prismoid is composed of prisms or prisms and wedges.

Hence, it is common to say that the volume computed using the End-Area formula is an estimate of the true volume.

Some surveying texts (e.g., Schofield 2002, Elfick, Fryer, Brinker & Wolf 1994) ascribe the difference between the Prismoidal and End-Area formulas to the presence of pyramids, which is true; and note that applying the End-Area formula to the computation of the volume of a pyramid produces a result that is larger than the correct value. This is also true; and these facts are used as justification for statements like:

> *The End-Area formula gives results that are generally larger than true volumes.*

Such statements could be misleading if the word *generally* was taken to mean *in all cases*. As can be seen in the volume computations in Table 1, the End-Area formula underestimates the true volume of Figures 3 and 5; correctly estimates the volume of Figure 2; and overestimates the volume of Figure 4.

The Prismoidal Correction (PC)

In earthwork volume computations in engineering surveying, it is common practice to compute volumes using the End-Area formula and then apply a correction to obtain a volume that would have been obtained if the Prismoidal formula – requiring additional field measurements for mid-section areas – was used. This leads to the definition of the *Prismoidal Correction* (PC) (Oliver & Clendinning 1978)

$$\begin{aligned} \text{PC} &= V_P - V_{EA} \\ &= \frac{L}{6}\left(A_1 + 4A_m + A_2\right) - \frac{L}{2}\left(A_1 + A_2\right) \\ &= -\frac{L}{6}\left(2A_1 - 4A_m + 2A_2\right) \\ &= -\frac{L}{3}\left(A_1 - 2A_m + A_2\right) \end{aligned} \tag{4}$$

If we compute the Prismoidal Correction (PC) given by equation (4) for each prismoid shown in Figures 2, 3, 4 and 5 using the values for A_1, A_m, A_2 in Table 1 and $L = 10$ m we obtain 0, $3\frac{1}{3}$ m^3, $-3\frac{1}{3}$ m^3 and $3\frac{1}{3}$ m^3. And, as expected, these are the differences between the two volumes (by Prismoidal formula and by End-Area formula) shown in Table 1.

But, common practice, as we have defined it, means mid-section areas are unknown, and hence a formula for the PC must be developed that is a function of end-areas only.

Prismoidal Correction for Three-Level road sections

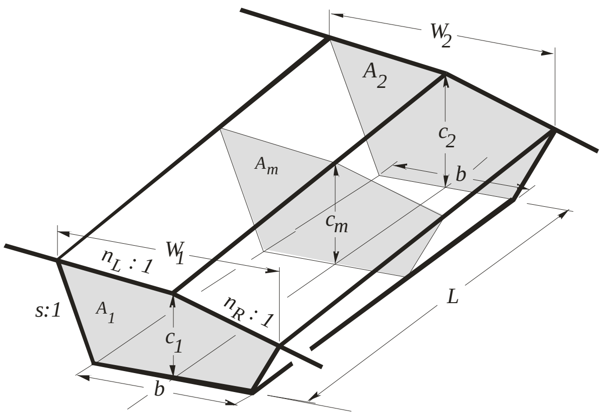


Figure 6: Cross-sectional view of a road in cut

Figure 6 is a cross-sectional view of a road in a cutting and Figure 7 shows typical cross-sections where the Three-Level Section (a) can be considered as the general type.

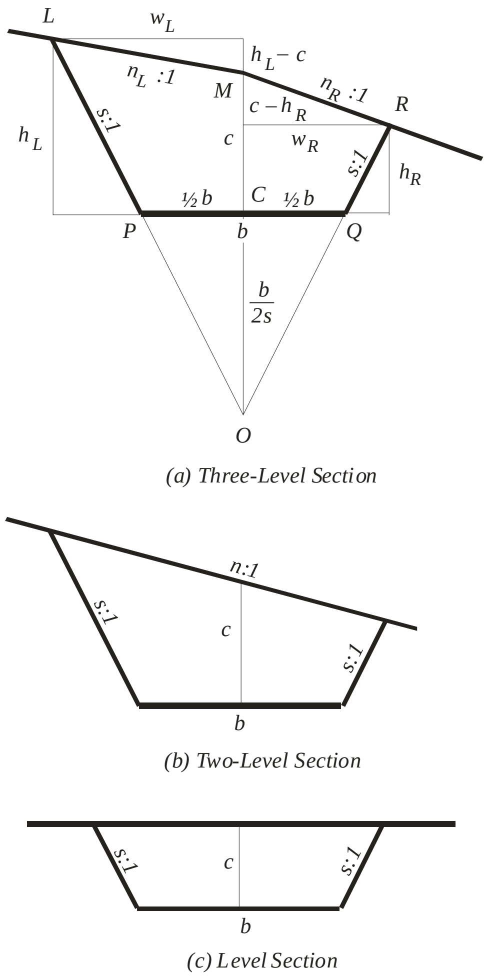


Figure 7: Three-Level, Two-Level and Level sections

The volume of the solid of length L can be computed once the end-section areas are known and these can be obtained from the basic design information; formation breadth *b;* side slopes of the cutting s horizontal to 1 vertical; transverse natural surface slopes, n_L, n_R horizontal to 1 vertical; depth of cut c at the formation centre-line; side-heights h_L, h_R and side-widths w_L, w_R where the subscripts L and R denote left and right.

Following Oliver & Clendinning (1978), the cross section areas can be obtained by considering the Three-Level Section (a) of Figure 7. Extending the side slopes to meet at O, we have, in the triangle OLM

$$\tfrac{1}{2}b + sh_L = w_L = n_L\left(h_L - c\right) \tag{5}$$

giving

$$h_L = \frac{n_L c + \frac{1}{2}b}{n_L - s} \tag{6}$$

and substituting back into equation (5) gives

$$w_L = \frac{n_L s}{n_L - s}\left(c + \frac{b}{2s}\right) \tag{7}$$

Similarly, with the right-hand triangle *OMR* we obtain

$$h_R = \frac{n_R c - \frac{1}{2}b}{n_R + s} \tag{8}$$

and
$$w_R = \frac{n_R s}{n_R + s}\left(c + \frac{b}{2s}\right) \tag{9}$$

For the Two-Level Section (b) and the Level Section (c) appropriate simplifications can be made to equations (6) to (9) and these are summarized in Tables 2 and 3. These formulae are the same for embankments where it is only necessary to invert the diagrams in Figures 6 and 7.

From the Three-Level Section (a) of Figure 7, the area of the section is the area of the two triangles *OLM* and *OMR* less the constructed isosceles triangle *OPQ*. Hence, with $W = w_L + w_R$, the cross section area is

$$\begin{aligned} A &= \tfrac{1}{2} w_L\left(c + \frac{b}{2s}\right) + \tfrac{1}{2} w_R\left(c + \frac{b}{2s}\right) - \frac{b^2}{4s} \\ &= \tfrac{1}{2} W\left(c + \frac{b}{2s}\right) - \frac{b^2}{4s} \end{aligned} \tag{10}$$

Section	side heights h_L, h_R
Three-Level	$h_L = \frac{n_L c + \frac{1}{2}b}{n_L - s}$ $\quad h_R = \frac{n_R c - \frac{1}{2}b}{n_R + s}$
Two-Level $n_L = n_R = n$	$h_L = \frac{nc + \frac{1}{2}b}{n - s}$ $\quad h_R = \frac{nc - \frac{1}{2}b}{n + s}$
Level	$h_L = h_R = c$

Table 2: Side heights for sections

Section	side widths w_L, w_R
Three-Level	$w_L = \frac{n_L s}{n_L - s}\left(c + \frac{b}{2s}\right)$ $\quad w_R = \frac{n_R s}{n_R + s}\left(c + \frac{b}{2s}\right)$
Two-Level $n_L = n_R = n$	$w_L = \frac{ns}{n - s}\left(c + \frac{b}{2s}\right)$ $\quad w_R = \frac{ns}{n + s}\left(c + \frac{b}{2s}\right)$
Level	$w_L = w_R = \frac{1}{2}b + sc$

Table 3: Side widths for sections

Using equations (4) and (10) with section widths W_1, W_2 and $W_m = \frac{1}{2}(W_1 + W_2)$; depths of cut c_1, c_2 and $c_m = \frac{1}{2}(c_1 + c_2)$ we obtain the Prismoidal Correction (PC) as

$$PC = -\frac{L}{12}(W_1 - W_2)(c_1 - c_2) \tag{11}$$

Prismoidal Correction for Side-Hill road sections

Figure 8 is a cross-sectional view of a road partly in cut and partly in fill and Figure 9 shows typical Side-Hill sections. The volume of the solid of length *L* can be computed once the end-section areas of cut and fill are known.

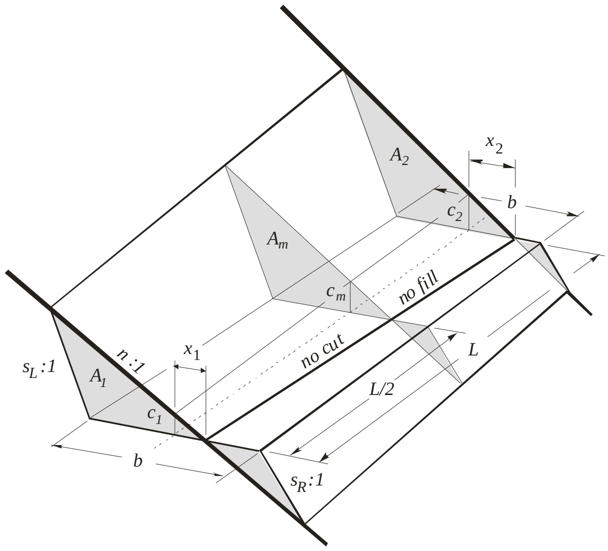


Figure 8: Cross-sectional view of road partly in cut and partly in fill

Using similar methods as before but with different side slopes s_L, s_R horizontal to 1 vertical; a single transverse slope of n horizontal to 1 vertical and side-heights denoted d_L, d_R. For the Side-Hill Section (a) of Figure 9 where <u>c is in cut</u>, we obtain, for the left-hand side of the section

$$d_L = \frac{\frac{1}{2}b + nc}{n - s_L} \tag{12}$$

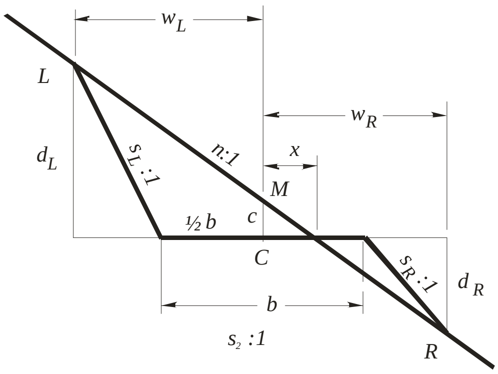
$$w_L = \frac{ns_L}{n - s_L}\left(\frac{b}{2s_L} + c\right) \tag{13}$$

Similarly, on the right-hand side we obtain

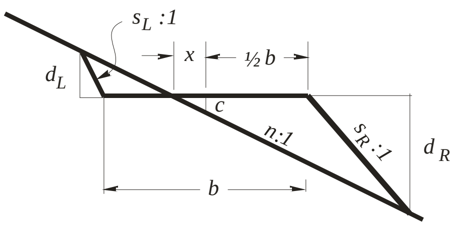
$$d_R = \frac{\frac{1}{2}b - nc}{n - s_R} \tag{14}$$

$$w_R = \frac{ns_R}{n - s_R}\left(\frac{b}{2s_R} - c\right) \tag{15}$$

For Side-Hill Sections where <u>c is in fill</u> appropriate changes can be made to equations (12) to (15) and these are summarized in Tables 4 and 5.



(a) Side-Hill Section
c in cut



(b) Side-Hill Section
c in fill

Figure 9: Side-Hill sections

Section	side heights h_L, h_R
Side-Hill c in cut	$d_L = \frac{\frac{1}{2}b + nc}{n - s_L}$ $\quad d_R = \frac{\frac{1}{2}b - nc}{n - s_R}$
Side-Hill c in cut	$d_L = \frac{\frac{1}{2}b - nc}{n - s_L}$ $\quad d_R = \frac{\frac{1}{2}b + nc}{n - s_R}$

Table 4: Side heights for Side-Hill sections

Section	side widths w_L, w_R
Side-Hill c in cut	$w_L = \frac{ns_L}{n - s_L}\left(\frac{b}{2s_L} + c\right)$ $w_R = \frac{ns_R}{n - s_R}\left(\frac{b}{2s_R} - c\right)$
Side-Hill c in cut	$w_L = \frac{ns_L}{n - s_L}\left(\frac{b}{2s_L} - c\right)$ $w_R = \frac{ns_R}{n - s_R}\left(\frac{b}{2s_R} + c\right)$

Table 5: Side widths for Side-Hill sections

With $x = nc$ denoting a distance from the centre-line to the no-cut/no-fill line (the point where the natural surface intercepts the road formation), the areas in cut and fill for the Side-Hill Section (a) of Figure 10 where <u>c is in cut</u> are

$$\begin{aligned} A_{CUT} &= \tfrac{1}{2}d_L\left(\tfrac{1}{2}b + x\right) \\ A_{FILL} &= \tfrac{1}{2}d_R\left(\tfrac{1}{2}b - x\right) \end{aligned} \tag{16}$$

Using equations (4) and (16) with x_1, x_2 denoting distances from the centre-line to the no-cut/no-fill line at the end-sections and $d_{1L}, d_{1R}, d_{2L}, d_{2R}$ denoting left and right side-heights at the end-sections we obtain the Prismoidal Corrections for cut and fill where <u>c is in cut</u>

$$\begin{aligned} PC_{CUT} &= -\frac{L}{12}\left(d_{1L} - d_{2L}\right)\left(x_1 - x_2\right) \\ PC_{FILL} &= \frac{L}{12}\left(d_{1R} - d_{2R}\right)\left(x_1 - x_2\right) \end{aligned} \tag{17}$$

Where <u>c is in fill</u>, the areas of cut and fill are

$$\begin{aligned} A^*_{CUT} &= \tfrac{1}{2}d_L\left(\tfrac{1}{2}b - x\right) \\ A^*_{FILL} &= \tfrac{1}{2}d_R\left(\tfrac{1}{2}b + x\right) \end{aligned} \tag{18}$$

leading to another pair of Prismoidal Corrections where <u>c is in fill</u>

$$\begin{aligned} PC^*_{CUT} &= \frac{L}{12}\left(d_{1L} - d_{2L}\right)\left(x_1 - x_2\right) \\ PC^*_{FILL} &= -\frac{L}{12}\left(d_{1R} - d_{2R}\right)\left(x_1 - x_2\right) \end{aligned} \tag{19}$$

Summary of Prismoidal Corrections

For volumes V estimated by the End-Area formula with Prismoidal Correction (PC)

$$V = \frac{L}{2}\left(A_1 + A_2\right) + PC \qquad (20)$$

the Prismoidal Corrections, applicable to road cross-sections can be summarised as

<table>
<tr><td>Cross-section</td><td>Prismoidal Correction (PC)</td></tr>
<tr><td>Three-Level</td><td rowspan="3">$PC = -\frac{L}{12}\left(W_1 - W_2\right)\left(c_1 - c_2\right)$</td></tr>
<tr><td>Two-Level</td></tr>
<tr><td>Level</td></tr>
<tr><td>Side-Hill
c in cut</td><td>$PC_{CUT} = -\frac{L}{12}\left(d_{1L} - d_{2L}\right)\left(x_1 - x_2\right)$
$PC_{FILL} = \frac{L}{12}\left(d_{1R} - d_{2R}\right)\left(x_1 - x_2\right)$</td></tr>
<tr><td>Side-Hill
c in fill</td><td>$PC^*_{CUT} = \frac{L}{12}\left(d_{1L} - d_{2L}\right)\left(x_1 - x_2\right)$
$PC^*_{FILL} = -\frac{L}{12}\left(d_{1R} - d_{2R}\right)\left(x_1 - x_2\right)$</td></tr>
</table>

Table 6: Prismoidal Corrections for road cross-sections

We can see here that none of these Prismoidal Corrections is appropriate for any of the prismoids in Figures 2, 3, 4 and 5.

The Prismoidal formula and curved surfaces

It is interesting to note that the Prismoidal formula can be applied to certain solids of revolution, e.g., sphere and ellipsoid to obtain formula for their volumes. And Simpson (1743) extended his rule for the computation of areas under a parabolic curve (his rule was originally an invention of Sir Isaac Newton) to develop a formula for the volume of frustums generated by rotating conic sections about their axes. We would recognise his formula as equation (1). But, applying the Prismoidal formula to other solids of revolution does not necessarily give the correct volume.

Figure 10 shows a barrel formed from the middle frustum of a spindle. It is either a *parabolic barrel* if it is the middle frustum of a parabolic spindle generated by the rotation of a parabola *ECF* about the line *AB;* or a *circular barrel* if it is the middle frustum of a circular spindle formed by the rotation of the segment of a circle *ACB* about its chord *AB.*

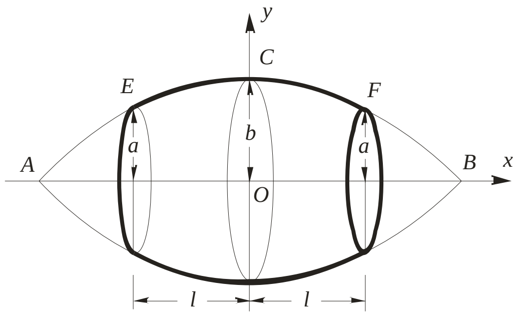


Figure 10: Barrel formed from middle frustum of a spindle

In the case of a parabolic barrel the formula for the volume is obtained by integration as

$$\begin{aligned} V &= 2\int_0^l \pi y^2 dx \\ &= \frac{2\pi l}{15}\left(8b^2 + 4ab + 3a^2\right) \\ &= \frac{\pi L}{15}\left(8b^2 + 4ab + 3a^2\right) \end{aligned} \qquad (21)$$

where $L = 2l$ and where the equation of the parabola *ECF* is

$$y = b - \frac{b-a}{l^2}x^2 \quad \text{for} \quad -l \le x \le l \qquad (22)$$

Applying the Prismoidal formula gives the volume of the parabolic barrel as

$$V_P = \frac{\pi L}{3}\left(2b^2 + a^2\right) = \frac{\pi L}{15}\left(10b^2 + 5a^2\right) \qquad (23)$$

And using equations (21) and (23) the difference $V - V_P$ is

$$\begin{aligned} V - V_P &= \frac{\pi L}{15}\left(8b^2 + 4ab + 3a^2\right) - \frac{\pi L}{15}\left(10b^2 + 5a^2\right) \\ &= -\frac{2\pi L}{15}\left(b-a\right)^2 \end{aligned} \qquad (24)$$

So $V < V_P$ since $b > a$ or the volume computed by the Prismoidal formula is an overestimation of the true volume of the parabolic barrel.

For the case of a circular barrel, the equation of the circle passing through the three points $E\left(-l,a\right)$, $C\left(0,b\right)$ and $F\left(l,a\right)$ of Figure 10 is

$$U\left(x^2 + y^2\right) + Wy = 1 \qquad (25)$$

where $U = \dfrac{b-a}{b\left(a^2+l^2-ab\right)}$ and $W = \dfrac{a^2+l^2-b^2}{b\left(a^2+l^2-ab\right)}$.

Completing the square in y in equation (25), re-arranging and taking the positive square-root gives

$$y = \sqrt{P^2 - x^2} - \frac{W}{2U} \quad \text{for} \quad -l \le x \le l \qquad (26)$$

and so
$$y^2 = Q - x^2 - \frac{W}{U}\sqrt{P^2 - x^2} \qquad (27)$$

where $P^2 = \dfrac{1}{U} + \left(\dfrac{W}{2U}\right)^2$ and $Q = 2P^2 - \dfrac{1}{U}$.

The volume of the barrel formed by rotating the circular arc *ECF* about the line *AB* is then

$$\begin{aligned}
V &= 2\int_0^l \pi y^2 dx \\
&= 2\pi \int_0^l \left(Q - x^2 - \frac{W}{U}\sqrt{P^2 - x^2}\right) dx \\
&= 2\pi \left(Ql - \frac{1}{3}l^3 - \frac{Wl}{2U}\sqrt{P^2 - l^2} - \frac{WP^2}{2U}\arcsin\frac{l}{P}\right) \\
&= \pi L \left(Q - \frac{L^2}{12} - \frac{W}{4U}\sqrt{4P^2 - L^2} - \frac{WP^2}{LU}\arcsin\frac{L}{2P}\right) \qquad (28)
\end{aligned}$$

And using equations (23) and (28) the difference $V - V_P$ is

$$\begin{aligned}
V - V_P &= -\pi L \left(\frac{L^2}{12} + \frac{W}{4U}\sqrt{4P^2 - L^2}\right. \\
&\qquad \left. + \frac{WP^2}{LU}\arcsin\frac{L}{2P} + \frac{2b^2 + a^2}{3} - Q\right) \qquad (29)
\end{aligned}$$

Assuming $b > a$ and $l > \sqrt{b^2 - a^2}$ ensures that $\dfrac{W}{U} > 0$, $P > 0$ and $4P^2 - L^2 > 0$; and it can be shown that $V - V_P > -\dfrac{\pi WP^2}{U}\arcsin\dfrac{L}{2P}$. But $\dfrac{\pi WP^2}{U}\arcsin\dfrac{L}{2P} > 0$ so it cannot be concluded that $V - V_P < 0$ nor that $V - V_P > 0$.

Note that if $l^2 = b^2 - a^2$, the spindle from which the barrel is formed is a sphere of radius *b;* and $V - V_P = 0$, i.e., the volume computed by the Prismoidal formula gives the exact true volume *V*.

How can these estimation errors be explained? Consider the volume of the frustum of the paraboloid *AEFB* of Figure 11 where it is assumed that the curve *AOB* is a parabola having the general equation

$$y = \sqrt{cx} \qquad (30)$$

where *c* is a constant.

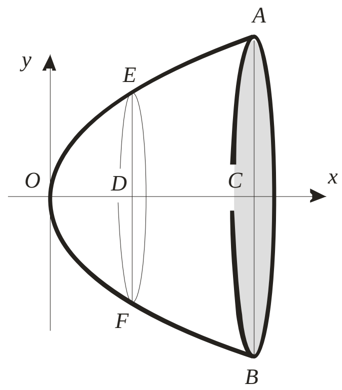


Figure 11: Paraboloid

The volume of the frustum between $p = OD$ and $q = OC$ is

$$V = \int_p^q \pi y^2 dx = \int_p^q \pi cx\, dx = \frac{c\pi}{2}\left(q^2 - p^2\right) \qquad (31)$$

The volume of this frustum using the Prismoidal formula [equation (1)] where $A_1 = \pi y_1^2 = \pi cp$, $A_2 = \pi y_2^2 = \pi cq$ and $4A_m = 4\pi y_m^2 = 4\pi c\left(\dfrac{p+q}{2}\right) = 2\pi c\left(p+q\right)$ is

$$\begin{aligned}
V_P &= \frac{L}{6}\left(A_1 + 4A_m + A_2\right) \\
&= \frac{\left(q-p\right)}{6} c\pi\left(3p + 3q\right) \\
&= \frac{c\pi}{2}\left(q^2 - p^2\right) \qquad (32)
\end{aligned}$$

And $V = V_P$ as Simpson (1743) proved. We may write this equivalence $V = V_P$ as *Simpson's rule*

$$\int_p^q f\left(x\right) dx = \frac{q-p}{6}\left(f\left(p\right) + 4f\left(\frac{p+q}{2}\right) + f\left(q\right)\right) + C \qquad (33)$$

where *C* is a constant and the integral on the left-hand-side of equation (33) is the volume generated by revolving the curve $y = f\left(x\right)$ about the *x*-axis between the lines $x = p$ and $x = q > p$.

It is known that $C = 0$ if $f(x)$ are polynomials of degree ≤ 3; i.e., linear, quadratic or cubic functions having the general forms: $y = cx + d$, $y = bx^2 + cx + d$ and $y = ax^3 + bx^2 + cx + d$ respectively (Apostol 1969). For any other function $f(x)$ on the left-hand-side of equation (33), C will not equal 0 on the right-hand-side, and so can be regarded as a correction to the volume computed by the Prismoidal formula to obtain the correct volume.

This explains why the volumes of certain solids of revolution can be correctly evaluated by using the Prismoidal formula and others cannot. For example:

(a) Frustum of a paraboloid formed by rotating the parabola $y = \sqrt{cx}$ about its axis of symmetry.

$V = \int_p^q \pi y^2 dx = \int_p^q f(x)\, dx$ and $f(x) = \pi cx$ is a linear function. And so the Prismoidal formula will give the correct volume.

(b) Barrel formed from the middle frustum of a parabolic spindle (see Figure 10) where the equation of the parabola is $y = b - \frac{b-a}{l^2}x^2$ [see equation (22)] and $V = 2\int_0^l \pi y^2 dx = 2\int_0^l f(x)\, dx$ and $f(x) = \frac{\pi(b-a)^2}{l^4}x^4 - \frac{2\pi b(b-a)}{l^2}x^2 + \pi b^2$ which is a polynomial in x of degree >3. And so the Prismoidal formula will not give the correct volume as is demonstrated by equation (24).

(c) Barrel formed from the middle frustum of a circular spindle (see Figure 10) where the equation of the circular arc is $y = \sqrt{P^2 - x^2} - \frac{W}{2U}$ [see equation (26)] and

$V = 2\int_0^l \pi y^2 dx = 2\int_0^l f(x)\, dx$. With equation (27)

$$f(x) = \pi\left(Q - x^2 - \frac{PW}{U}\sqrt{1 - \left(\frac{x}{P}\right)^2}\right)$$

and, using the binomial series

$$\sqrt{1 - \left(\frac{x}{P}\right)^2} = 1 - \frac{1}{2}\frac{x}{P} - \frac{1}{2\cdot 4}\left(\frac{x}{P}\right)^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{x}{P}\right)^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\left(\frac{x}{P}\right)^4 - \cdots$$

Thus $f(x)$ will be a polynomial in x of degree >3. And so the Prismoidal formula will not give the exact volume.

Now, imagine the parabolic barrel of Figure 10 trimmed of its sides and bottom leaving a solid that could be likened to a loaf of bread – a square prism topped with a paraboloidal cap. The volume of this solid will be some portion of the original barrel, and noting (b) above, we may express the volume of the loaf of bread as $V = 2B\int_0^l \pi y^2 dx = 2B\int_0^l f(x)\, dx$ where B will be less than one and $f(x)$ remains unaltered; a polynomial in x of degree >3. Hence the Prismoidal formula will not give the correct volume of the loaf of bread.

We can use these examples of solids of revolution, or portions thereof, in support of a statement:

> *Beware of using the Prismoidal formula for estimating the volume of solids having curved faces; the volume may not be correct.*

Prismoidal formula and Finite-Element volumes

Davis (1994) introduces the Finite-Element-Volume method of computing earthwork volumes. The method is designed to overcome the limitations of computing volumes by the End-Area formula and can be applied to straight or curved road alignments. For the purposes of numerical comparison of finite-element volumes of certain alignments (with triangular cross-sections) with those obtained by conventional methods he gives a general curvilinear volume formula as

$$V = \int_0^L A(1 + ek)\, dx = \int_0^L A\, dx + \int_0^L Aek\, dx \qquad (34)$$

A is the cross-sectional area of a differential element of volume, e is the eccentricity of the centroid of the cross-section, i.e., the distance from the curved centreline to the centroid and k is the curvature of the centreline. A, e and k are functions of the centreline length x and Davis (ibid.) defines the second term on the right-hand side of (34) as a *prismoidal curvature correction* V_C where the term "prismoidal" relates to the fact that the stated formula for V_C are Simpson's rule approximations of the integral $\int_0^L Aek\, dx$. The definition of prismoidal correction that we use in this paper is different [see equation (4)].

Davis (ibid.) does not use a "prismoidal correction" in the computation of finite-element volumes – which can be outlined in the following way.

The length x of the solid section is divided into n small slices, each of width δx, and with the *j-th* slice having cross-sectional area A_j. The finite-element volume is then

$$V_{FE} = \sum_{j=1}^{n} A_j \, \delta x \qquad (35)$$

In the limit as $n \to \infty$ and $\delta x \to 0$, V_{FE} approaches the "true" volume *V*.

For curved alignments the section length is adjusted to equal the path length of the centroids of the cross-sections. And where the plane end-faces of the section have different numbers of vertices; then certain rules for "tapering" are adopted.

Interestingly, if this method is used to compute the volume of the prismoid shown in Figure 3, the resulting volume would be an estimate of a volume of a solid having a curved upper-face as shown in Figure 12.

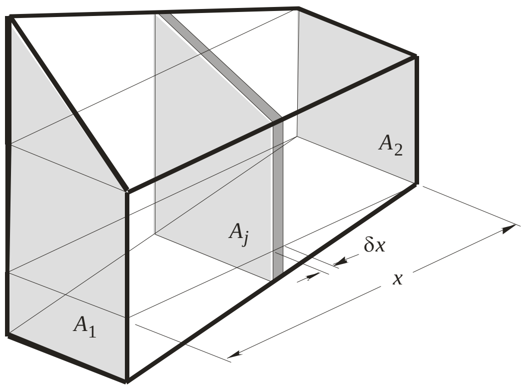


Figure 12: Finite-element volume

Pyramid frustum formula

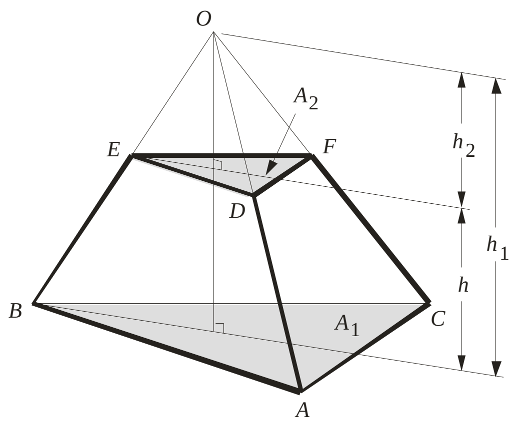


Figure 13: Frustum of a triangular pyramid

For earthwork volume computations where cross-sections change from cut to fill, or vice versa, some authors, e.g., Easa 1991, Moffitt and Boucher 1987, suggest that volumes of these "transition areas" can be more accurately estimated by the *Pyramid Frustum* formula rather than the End-Area formula.

The Pyramid Frustum volume formula can be derived in the following manner. From similar triangles (end-faces and side-faces of Figure 13) the following ratios may be obtained

$$\frac{h_1}{h_2} = \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} \qquad (36)$$

and

$$\frac{A_1}{A_2} = \frac{(AB)^2}{(DE)^2} = \frac{(BC)^2}{(EF)^2} = \frac{(CA)^2}{(FD)^2} \qquad (37)$$

From equations (36) and (37) we obtain

$$\frac{\sqrt{A_1}}{\sqrt{A_2}} = \frac{h_1}{h_2} = \frac{h + h_2}{h_2} = \frac{h}{h_2} + 1$$

which can be manipulated to yield

$$\frac{h_2}{h} = \frac{\sqrt{A_1 A_2} + A_2}{A_1 - A_2} \qquad (38)$$

Now the volume of the frustum *ABCDEF* of Figure 13 is the difference of the two pyramids *OABC* and *ODEF*, or

$$\begin{aligned} V_{PF} &= \tfrac{1}{3} A_1 (h + h_2) - \tfrac{1}{3} A_2 h_2 \\ &= \frac{h}{3}\left(A_1 + (A_1 - A_2)\frac{h_2}{h} \right) \end{aligned} \qquad (39)$$

And substituting equation (38) into equation (39) gives the Pyramid Frustum volume formula

$$V_{PF} = \frac{h}{3}\left(A_1 + \sqrt{A_1 A_2} + A_2 \right) \qquad (40)$$

Whilst this formula has been developed for the frustum of a triangular pyramid it is applicable to frustums of other pyramids, since they could all be decomposed into frustums of triangular pyramids, each having a common edge which could be expended to the apex of the pyramid.

The volumes shown in Table 7 indicate that the Pyramid Frustum formula is no better than the End-Area formula in estimating volumes of the prismoids of Figures 2 to 5.

Prismoid	Areas m^2			Volumes m^3		
	A_1	A_m	A_2	V_P	V_{EA}	V_{PF}
Figure 2	6	5.0	4	50	50	49.67
Figure 3	8	6.5	4	$63\frac{1}{3}$	60	58.86
Figure 4	2	2.5	4	$26\frac{2}{3}$	30	29.43
Figure 5	3	3.0	2	$28\frac{1}{3}$	25	24.83

Table 7: Volumes of prismoids

Conclusion

In this paper, we have provided a verification of the Prismoidal formula as enunciated by Sir Isaac Newton; its historical connection with *Simpson's rule* and its indirect usage in earthwork computations via the End Area formula and a Prismoidal Correction. Hopefully, we have demonstrated that there are a number of possible Prismoidal Corrections and not a single one that suits all solids.

In addition, some useful information regarding the limitations of the Prismoidal formula is presented to give a better appreciation of its use when the solid under consideration has curved faces.

We have also given a very brief outline of the finite-element method of computing volumes – a method suitable for solids with warped faces approximating the natural surface – and a development of the Pyramid Frustum formula. The Pyramid Frustum formula is sometimes used as a "better" estimate of the volume than the End-Area formula, but, as we have shown in our limited study, perhaps not in every case.

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